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Probability & Statistics,

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Poisson Distribution

Poisson Distribution: A random variable X is said to have a Poisson distribution with parameter λ, if its density function is given by:

$$\boxed{f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \text{ for } x = 0, 1, 2, \ldots. \qquad \lambda > 0.}$$

Poisson Distribution

To verify *P*(*S*) = 1 for Poisson distribution formula

$$\sum_{x=0}^{\infty} f(x;\lambda) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^{x}}{x!}$$

Since the infinite series in the expression on the right is Maclaurin's series for e^{λ}, it follows that

$$\sum_{x=0}^{\infty} f(x;\lambda) = e^{-\lambda}e^{\lambda} = 1.$$

Poisson Distribution

Mean and Variance

$$E(X)=\sum_{x=0}^{\infty} x f(x)$$

$$E(X)=\sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^{x}}{x!} \qquad =\lambda\sum_{x=1}^{\infty}\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$

$$E(X)=\lambda=\mu$$

Poisson Distribution

$$E(X^2)=\sum_{x=0}^{\infty} x^2 f(x)$$

$$E(X^2)=\sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad =\lambda\sum_{x=1}^{\infty} x\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$

$$=\lambda\sum_{x=1}^{\infty}(x-1+1)\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$

Poisson Distribution

$$= \lambda \sum_{x=1}^{\infty} \left[(x-1) \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!} + 1 \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!} \right]$$

$$= \lambda (\lambda + 1)$$

$$\sigma^2 = E(X^2) - E(X)^2 = \lambda$$

Poisson Distribution

Moment Generating function

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{tx} \lambda^x}{x!} e^{-\lambda}$$

Poisson Distribution

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{z^x}{x!}$$

where

$$z = e^t \lambda$$

$$= e^z e^{-\lambda}$$

$$M_X(t) = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

Example-1

The number of traffic accidents per week in a small city has a Poisson distribution with mean equal to 1.3. What is the probability of at least two accidents in the next week?

Poisson Processes

Suppose we are concerned with discrete events taking place over continuous intervals (not in the usual mathematical sense) of time, length or space; such as

- the arrival of telephone calls at a switchboard
- number of red blood cells in a drop of blood (here the continuous interval involved is a drop of blood).
- number accidents in a city per year.

Steps for solving Poisson process problems

- Determine the average number of occurrences of the event per unit (i.e. λ).
- Determine the length or size of the interval (i.e. s).
- The random variable X, the number of occurrences of the event in the interval of size s follows a Poisson distribution with parameter $k = \lambda s$.

i.e.,

$$\boxed{f(x) = \frac{e^{-\lambda s}(\lambda s)^x}{x!} \text{ for } x = 0, 1, 2, \ldots.}$$

Example-2

The number of traffic accidents per week in a small city has a Poisson distribution with mean equal to 3. What is the probability of at least one accidents in the next 2 weeks?

Ans:0.9975

Example-3

Suppose flaws (cracks, chips, specks, etc.) occur on the surface of glass with density of 3 per square meter. What is the probability of there being exactly 4 flaws on a sheet of glass of area 0.5 square meter?

Ans: 047

Example-4

The arrival of trucks at a receiving dock is a Poisson process with a mean arrival rate of 2 per hour.

(a) Find the probability that exactly 5 trucks arrive in a two hour period.

(b) Find the probability that 8 or more truck arrive in a two hour period.

(c) Find the probability that exactly two trucks arrive in a one hour period and exactly 3 trucks arrive in the next one hour period.

Solution: Given $\lambda = 2$ trucks/hr,

(a) $s = 2 \Rightarrow k = 4$

$P(X = 5) = 0.156$

(b) $P(X \geq 8) = 1 - F(7;4) = 1 – 0.949 = 0.051$

(c) For first 1 hour period the probability is

$f(2;2) = 2^2.e^{-2}/2! = 0.2707$

and second 1 hour period the probability is

$f(3;2) = 2^3.e^{-2}/3! = 0.1804$

These two intervals do not overlap so the counts are independent, hence required probability

$= f(2;2).\ f(3;2) = (0.2707)(0.1804) = 0.0488$

Poisson Approximation to the Binomial Distribution

To Show that when $n \to \infty$ and $p \to 0$, while $np = \lambda$ remain constant, $b(x;n,p) \to f(x;\lambda)$.

Proof: First we substitute λ/n for p into the formula for the binomial distribution, we get

$$b(x;n,p) = b(x;n,\frac{\lambda}{n}) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

Poisson Approximation to the Binomial Distribution

$$b(x; n, p) = \frac{n(n-1)(n-2)\ \dots\ (n-x+1)}{x!\,n^x}\lambda^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{x-1}{n}\right)}{x!}\lambda^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

If $n \to \infty$, we have

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{x-1}{n}\right) \to 1$$

Poisson Approximation to the Binomial Distribution

and $$\left(1-\frac{\lambda}{n}\right)^{n-x}=\left[\left(1-\frac{\lambda}{n}\right)^{n}\right]\left(1-\frac{\lambda}{n}\right)^{-x}\to e^{-\lambda}.$$

Hence

$$b(x;n,p)\to\frac{\lambda^{x}e^{-\lambda}}{x!}=f(x;\lambda)\ \text{for}\ x=0,1,2,\ldots.$$

Poisson Approximation to the Binomial Distribution

Note:

An acceptable rule of thumb is to use Poisson approximation to the binomial distribution

- if $n \geq 20$ and $p \leq 0.05$;
- if $n \geq 100$, the approximation is generally excellent so long as $np \leq 10$.

Poisson Approximation to the Binomial Distribution

Comparison of Poisson and binomial probabilities

Example: It is known that 5% of the books bound at a certain bindery have defective binding. Find the probability that 2 of 100 books bound by this bindery will have defective binding using:

(a) the formula for the binomial distribution;

(b) the Poisson approximation to the binomial distribution.

Poisson Approximation to the Binomial Distribution

Solution: Given $x = 2$, $n = 100$ and $p = 0.05$,

$\lambda = n.p = 5$

$$\text{(a) } b(x;100,0.05) = \binom{100}{2}(0.05)^2(0.95)^{98} = 0.081$$

$$\text{(b) } f(2;5) = \frac{5^2.e^{-5}}{2!} = 0.084$$

The difference between the two values we obtained is only 0.003. When we use Table 2,

$f(2; 5) = F(2; 5) - F(1; 5) = 0.125 – 0.040 = 0.085$.